The Method of Addition for Deriving Continuous Withdrawal Equations

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The verified gravity corrected theory for continuous withdrawal of flat plates from Newtonian liquids (5), when expressed in the speed-explicit form (2) becomes

$$C_o = 1.09\ D^{3/2} + D^2 \quad (1)$$

where

$$C_o = u_w\mu/\sigma \quad (1a)$$

and

$$D = h(\rho g/\sigma)^{1/2} \quad (1b)$$

When we rewrite the two other theories for this case in terms of C_o and D they become

Low Speed Theory: $C_o = 1.09\ D^{3/2}$ (2)

Medium Speed Theory: $C_o = D^2$ (3)

Equation (1) was derived from a solution of a third-order differential equation. The derivations of Equations (2) and (3)are reviewed elsewhere (4).

In hindsight, it appears that Equation (1) could have been developed from Equations (2) and (3) by addition of the terms on the right hand side. The development of Equation (1) by addition is a simple example of the method of addition.

The purposes of this paper are to present this new method of addition and to develop a new theory with the method. The new theory is one for the case of continuous withdrawal of cylinders from Ellis fluids.

FLAT PLATE THEORY FOR NON-NEWTONIAN FLUIDS

The theory for withdrawal of flat plates from Ellis fluids (3) may be written in speed-explicit form as

$$C_o = 1.09\ D^{3/2} + D^2 + 1.09\left(\frac{3\alpha}{\alpha+2}\right)(B)\ D^{\alpha+0.5} + (B)\ D^{\alpha+1} \quad (4)$$

where

$$-\frac{du}{dy} = |a_0 + a_1{}^\alpha\, \tau^{\alpha-1}|\tau \quad (4a)$$

$$C_o = u_w/a_0\sigma \quad (4b)$$

$$B = \frac{[a_1(\rho g\sigma)^{1/2}]^\alpha}{[a_0(\rho g\sigma)^{1/2}]} \quad (4c)$$

The form of Equation (4) is new. By comparison of the Ellis theory with Newtonian theory in the speed-explicit, D form [Equations (1) and (4)], it is clear that the first two terms in Equation (4) represent the Newtonian contribution. The earlier speed-explicit form (2) contained a fluid property group, A, where

$$A = \frac{(a_1\sigma/h)^\alpha}{(a_0\sigma/h)} \quad (4d)$$

The thickness, h, was eliminated from the fluid property parameter, A, by noting that $B = AD^{\alpha-1}$.

A theory for power law fluids follows from Equation (4); by neglecting the Newtonian terms, we obtain

$$C_o = 1.09\left(\frac{3\alpha}{\alpha+2}\right)(B)\ D^{\alpha+0.5} + (B)\ D^{\alpha+1} \quad (5)$$

The agreement of Equation (5) with data is better than that obtained with the two previous power law theories given by Equations (41) and (52) of an earlier paper (1). Equation (52) (1) may be rewritten as

$$C_o = 1.09\left(\frac{0.642}{P}\right)^{3/2}\left(\frac{3\alpha}{\alpha+2}\right)(B)\ (D)^{\alpha+0.5} + (B)\ D^{\alpha+1} \quad (5a)$$

where $P = P(\alpha) \geqq 0.642$ as noted in Table 2 of the earlier paper (1). The two-term theory of Equation (5a) reduces to Equation (5) for $P = 0.642$.

CYLINDER THEORY FOR NEWTONIAN FLUIDS

The verified gravity corrected theory for cylinder withdrawal from Newtonian fluids (6) may be written in speed-explicit form as

$$C_o = 1.09\,[D\ C_m]^{3/2} + [G\sqrt{2Y}]^2 \quad (6)$$

where

$$G = R(\rho g/2\sigma)^{1/2} \quad (6a)$$

$$S = 1 + (h/R) \quad (6b)$$

$$Y = S^2 \ln S - 0.5\,(S^2 - 1) \quad (6c)$$

$$C_m = \frac{2.4(GS)^{0.85}}{1 + 2.4(GS)^{0.85}} + \frac{1}{2GS} \quad (6d)$$

Here C_m is the meniscus curvature which includes the simultaneous effect of the two principal radii of curvature; C_m was derived from an extension of static meniscus theory.

At large G, C_m approaches one and $2YG^2$ approaches D^2. For flat plates, therefore, Equation (6) reduces to Equation (1).

CYLINDER THEORY FOR NON-NEWTONIAN FLUIDS

We now derive a new theory by addition. If we modify power law Equation (5) by analogy with Equations (1) and (6) and add the resulting terms to Equation (6), we obtain a cylinder theory for Ellis fluids

$$C_o = 1.09\,[DC_m]^{3/2} + [G\sqrt{2Y}]^2 + 1.09\left(\frac{3\alpha}{\alpha+2}\right)(B)\ [DC_m]^{\alpha+0.5} + (B)\ [G\sqrt{2Y}]^{\alpha+1} \quad (7)$$

From Equation (7), we can also write a two-term theory for cylinder withdrawal from power law fluids

$$C_o = 1.09\left(\frac{3\alpha}{\alpha+2}\right)(B)\ [DC_m]^{\alpha+0.5} + (B)\ [G\sqrt{2Y}]^{\alpha+1} \quad (8)$$

Low speed and medium speed theories can be obtained as special cases of Equations (7) and (8).

The above new theories are ready for testing as soon as suitable continuous withdrawal data becomes available.

DISCUSSION

Equations (7) and (8) are new theories and the forms of Equations (3), (4), and (5) are new.

The derivation of the suitable dimensionless form of Equation (4) suggested the method of addition for deriving new withdrawal theories and led to the understanding of the method. The suitable dimensionless form simply requires that the equation be speed explicit on one side (such as C_o) and that the film thickness (h) be included only in the parameter D.

The speed-explicit form (2) was first developed in dimensional form for the flat plate Ellis theory equation. The original purpose for the development (2) was to simplify numerical evaluation of the functional relationship between h and u_w for a given fluid. The method of addition presented herein is an important but unexpected use of the speed-explicit form.

Although development of the method of addition was based on the recent development of the speed-explicit form of withdrawal equations, it was not until after the speed-explicit paper (2) had been accepted that the suitable dimensionless form [Equation (4)] of the Ellis theory was derived.

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NOTATION

A = fluid property and thickness, Equation (4d)
a_0 = Ellis low shear parameter ($1/\mu$), Equation (4a)
a_1 = Ellis parameter, Equation (4a)
B = fluid properties, Equation (4c)
C_m = meniscus curvature, Equation (6d)
C_o = withdrawal speed parameter, $u_w(\mu/\sigma)$ or $u_w/(a_0\sigma)$
D = film thickness parameter, $h(\rho g/\sigma)^{1/2}$
G = radius parameter, $R(\rho g/2\sigma)^{1/2}$
g = acceleration of gravity, cm./sq.sec.
h = film thickness, cm.
P = curvature coefficient, Equation (5a)
R = cylinder radius, cm.
S = film thickness parameter, $1 + (h/R)$
u = fluid velocity, cm./sec.
u_w = withdrawal velocity of the solid, cm./sec.
Y = Y function, Equation (6c)
y = coordinate, cm.

Greek letters

α = Ellis exponent parameter, Equation (4a)
μ = viscosity, poise
ρ = density, g./cc.
σ = surface tension, dyne/cm.
τ = viscous shear stress, dyne/sq.cm.

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Some Properties of the Apparent Water Paradox in Entrainment

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Two entrainment phenomena of engineering importance are withdrawal and removal. We use withdrawal to mean the slow, vertical, upward passage of an axisymmetric solid through the free surface of a wetting liquid in such a way that liquid contact is maintained between the solid and the bath. Removal is a related condition in which a short object is raised above the bath after withdrawal so that the entrained film separates from the liquid bath. Removal experiments and continuous withdrawal experiments have been reported (*1*, *7*) and discussed (*5*) elsewhere.

Substantial progress has been made in describing withdrawal. For example, a theory developed for the continuous withdrawal of cylinders has recently been verified using removal tests as well as continuous withdrawal experiments (*7*). Specifically, this theory was found valid for radii of 0.01 cm. to infinity and for a wide range of speeds. Furthermore, all fluids tested, except one, agreed with the theory. The exceptional fluid was water. The behavior of water was noted in the theory paper (*7*) but no data were given nor was any satisfactory explanation offered. The exceptional behavior of water was called the *water paradox* (*6*, *7*); it might now be named the *apparent water paradox* in entrainment.

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